

1

$$\begin{aligned}
 (1) \quad & \lim_{x \rightarrow \infty} x \sin(\sqrt{a^2x^2 + b} - ax) \\
 &= \lim_{x \rightarrow \infty} x \sin \frac{a^2x^2 + b - (ax)^2}{\sqrt{a^2x^2 + b} + ax} \\
 &= \lim_{x \rightarrow \infty} x \sin \frac{b}{\sqrt{a^2x^2 + b} + ax} \\
 &= \lim_{x \rightarrow \infty} \frac{bx}{\sqrt{a^2x^2 + b} + ax} \cdot \frac{\sin \frac{b}{\sqrt{a^2x^2 + b} + ax}}{\frac{b}{\sqrt{a^2x^2 + b} + ax}} \\
 &= \lim_{x \rightarrow \infty} \frac{b}{\sqrt{a^2 + \frac{b}{x^2}} + a} \cdot \frac{\sin \frac{b}{\sqrt{a^2x^2 + b} + ax}}{\frac{b}{\sqrt{a^2x^2 + b} + ax}} \\
 &= \frac{b}{a + a} \times 1 \\
 &= \frac{b}{2a} \quad (\text{答})
 \end{aligned}$$

$$\begin{aligned}
 (2) \quad & z = 1 + \sqrt{3}i = 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = 2\left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right), \\
 & w = \frac{1}{\sqrt{2}\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)} = \frac{1}{\sqrt{2}\left(\cos \frac{3}{4}\pi + i \sin \frac{3}{4}\pi\right)} \\
 & \quad \quad \quad = \frac{1}{\sqrt{2}} \left\{ \cos\left(-\frac{3}{4}\pi\right) + i \sin\left(-\frac{3}{4}\pi\right) \right\}
 \end{aligned}$$

であるから,

$$r = |z^8 w^7| = 2^8 \times (2^{-\frac{1}{2}})^7 = 2^{\frac{9}{2}} = 16\sqrt{2} \quad (\text{答})$$

$$\theta = \arg z^8 w^7 = 8 \times \frac{\pi}{3} + 7 \times \left(-\frac{3}{4}\pi\right) + 2n\pi = \left(2n - \frac{31}{12}\right)\pi \quad (n \text{ は整数})$$

$0 \leq \theta < 2\pi$ では

$$\theta = \frac{17}{12}\pi \quad (\text{答})$$

2

(1)

$$\begin{aligned}3n^3 + n &= 3(n^3 + 1) + n - 3 \\n^3 + 1 &= (n - 3)(n^2 + 3n + 9) + 28\end{aligned}$$

より

$$\begin{aligned}g &= \gcd(3n^3 + n, n^3 + 1) \\&= \gcd(n^3 + 1, n - 3) \\&= \gcd(n - 3, 28)\end{aligned}$$

は $28 = 2^2 \times 7$ の約数であるから，特に

$$g \neq 5$$

である。

(証明おわり)

(2) 14 は 28 の約数であることに注意すると，

$$g = \gcd(n - 3, 28) = 14$$

となる n の最小値は，

$$n - 3 = 14$$

より

$$n = 17 \quad (\text{答})$$

3

(1) a, b, c がすべて異なる確率 p は

$$p = \frac{5}{6} \times \frac{4}{6} = \frac{20}{36}$$

 $a + b + c = 9$, $a < b < c$ となるのは

$$(a, b, c) = (1, 2, 6), (1, 3, 5), (2, 3, 4)$$

のときであるから, a, b, c の順序も考えて, a, b, c が相異なり, 和が 9 となる確率 q は

$$q = \frac{3 \times 3!}{6^3} = \frac{3}{36}$$

求める条件付き確率は

$$\frac{q}{p} = \frac{3}{20} \quad (\text{答})$$

(2) $t = \cos x$ とおくと, 2倍角・3倍角の公式より

$$\begin{aligned} \cos 3x + 4 \cos 2x - 8 \cos x &= (4t^3 - 3t) + 4(2t^2 - 1) - 8t \\ &= 4t^3 + 8t^2 - 11t - 4 \end{aligned}$$

これを $f(t)$ とおくと,

$$f'(t) = 12t^2 + 16t - 11 = (6t + 11)(2t - 1)$$

t	-1	$\frac{1}{2}$	1
$f'(t)$	-	0	+
$f(t)$		↘ 極小 ↗	

$$f(-1) = -4 + 8 - 11 - 4 = -11$$

$$f\left(\frac{1}{2}\right) = 4\left(\frac{1}{2}\right)^3 + 8\left(\frac{1}{2}\right)^2 - 11 \cdot \frac{1}{2} - 4 = \frac{1 + 4 - 11 - 8}{2} = -7$$

$$f(1) = 4 + 8 - 11 - 4 = -3$$

 $\cos 3x + 4 \cos 2x - 8 \cos x$ の

$$\text{最大値は } f(1) = -3, \text{ 最小値は } f\left(\frac{1}{2}\right) = -7 \quad (\text{答})$$

である。

4

(1)

$$f'(x) = \frac{1 \cdot (x^2 + 1) - x \cdot 2x}{(x^2 + 1)^2} = \frac{-x^2 + 1}{(x^2 + 1)^2} = \frac{-(x+1)(x-1)}{(x^2 + 1)^2}$$

x	-1	1
$f'(x)$	- 0 +	0 -
$f(x)$	↘ 極小 ↗	極大 ↘

よって, $f(x)$ の

$$\text{極大値は } f(1) = \frac{1}{2}, \text{ 極小値は } f(-1) = -\frac{1}{2} \quad (\text{答})$$

(2)

$$\begin{aligned} V &= \int_0^{\sqrt{3}} \pi |f(x) - a|^2 dx \\ &= \pi \int_0^{\sqrt{3}} \{f(x)\}^2 dx - 2\pi a \int_0^{\sqrt{3}} f(x) dx + \sqrt{3} \pi a^2 \end{aligned}$$

ここで,

$$\begin{aligned} \int_0^{\sqrt{3}} \{f(x)\}^2 dx &= \int_0^{\sqrt{3}} \frac{x^2}{(x^2 + 1)^2} dx \\ &= \int_0^{\frac{\pi}{3}} \frac{\tan^2 \theta}{(\tan^2 \theta + 1)^2} (\tan \theta)' d\theta \\ &= \int_0^{\frac{\pi}{3}} \tan^2 \theta \cos^4 \theta \cdot \frac{1}{\cos^2 \theta} d\theta \\ &= \int_0^{\frac{\pi}{3}} \sin^2 \theta d\theta \\ &= \frac{1}{2} \int_0^{\frac{\pi}{3}} (1 - \cos 2\theta) d\theta \\ &= \frac{1}{2} \left[\theta - \frac{1}{2} \sin 2\theta \right]_0^{\frac{\pi}{3}} \\ &= \frac{1}{2} \left(\frac{\pi}{3} - \frac{\sqrt{3}}{4} \right) \end{aligned}$$

$$\begin{aligned} \int_0^{\sqrt{3}} f(x) dx &= \frac{1}{2} \int_0^{\sqrt{3}} \frac{2x}{x^2 + 1} dx \\ &= \frac{1}{2} \int_0^{\sqrt{3}} \frac{(x^2 + 1)'}{x^2 + 1} dx \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{2} \left[\log(x^2 + 1) \right]_0^{\sqrt{3}} \\
 &= \frac{1}{2} (\log 4 - \log 1) \\
 &= \log 2
 \end{aligned}$$

であるから,

$$V = \pi \left(\sqrt{3} a^2 - 2a \log 2 + \frac{\pi}{6} - \frac{\sqrt{3}}{8} \right) \quad (\text{答})$$

(3)

$$\begin{aligned}
 V &= \pi \left\{ \sqrt{3} \left(a^2 - \frac{2}{\sqrt{3}} a \log 2 \right) + \frac{\pi}{6} - \frac{\sqrt{3}}{8} \right\} \\
 &= \pi \left\{ \sqrt{3} \left(a - \frac{\log 2}{\sqrt{3}} \right)^2 - \frac{(\log 2)^2}{\sqrt{3}} + \frac{\pi}{6} - \frac{\sqrt{3}}{8} \right\}
 \end{aligned}$$

は

$$a = \frac{\log 2}{\sqrt{3}} \text{ のとき最小値 } \left(\frac{\pi}{6} - \frac{(\log 2)^2}{\sqrt{3}} - \frac{\sqrt{3}}{8} \right) \pi \quad (\text{答})$$

をとる。