

図チャレ 第131回(2012年8月)

楕円 $\frac{x^2}{4} + y^2 = 1$ を原点のまわりに回転させて点 $(1, 1)$ を通るようにするとき, この回転変換を表す行列を求めよ。

解答

楕円 $\frac{x^2}{4} + y^2 = 1$ と (点 $(1, 1)$ を通る) 円 $x^2 + y^2 = 2$ の交点は,

$$\left(\frac{2}{\sqrt{3}}, \frac{\sqrt{2}}{\sqrt{3}}\right), \left(-\frac{2}{\sqrt{3}}, \frac{\sqrt{2}}{\sqrt{3}}\right), \left(-\frac{2}{\sqrt{3}}, -\frac{\sqrt{2}}{\sqrt{3}}\right), \left(\frac{2}{\sqrt{3}}, -\frac{\sqrt{2}}{\sqrt{3}}\right)$$

である。

$$\frac{2}{\sqrt{3}} = \sqrt{2} \cos \alpha, \quad \frac{\sqrt{2}}{\sqrt{3}} = \sqrt{2} \sin \alpha \quad (0 < \alpha < \frac{\pi}{4})$$

とおくと, 題意の回転変換の回転角 θ は

$$\theta = \frac{\pi}{4} - \alpha, \quad \frac{\pi}{4} - (\pi - \alpha), \quad \frac{\pi}{4} - (\pi + \alpha), \quad \frac{\pi}{4} - (-\alpha)$$

であり, 求める行列は次の4つである。

$$\begin{aligned} & \begin{pmatrix} \cos \frac{\pi}{4} & -\sin \frac{\pi}{4} \\ \sin \frac{\pi}{4} & \cos \frac{\pi}{4} \end{pmatrix} \begin{pmatrix} \cos(-\alpha) & -\sin(-\alpha) \\ \sin(-\alpha) & \cos(-\alpha) \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2}}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & \frac{\sqrt{2}}{\sqrt{3}} \end{pmatrix} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 + \sqrt{2} & 1 - \sqrt{2} \\ -1 + \sqrt{2} & 1 + \sqrt{2} \end{pmatrix} \\ & \begin{pmatrix} \cos \frac{\pi}{4} & -\sin \frac{\pi}{4} \\ \sin \frac{\pi}{4} & \cos \frac{\pi}{4} \end{pmatrix} \begin{pmatrix} \cos(\pi - \alpha) & \sin(\pi - \alpha) \\ -\sin(\pi - \alpha) & \cos(\pi - \alpha) \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} -\frac{\sqrt{2}}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & -\frac{\sqrt{2}}{\sqrt{3}} \end{pmatrix} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 - \sqrt{2} & 1 + \sqrt{2} \\ -1 - \sqrt{2} & 1 - \sqrt{2} \end{pmatrix} \\ & -\frac{1}{\sqrt{6}} \begin{pmatrix} 1 + \sqrt{2} & 1 - \sqrt{2} \\ -1 + \sqrt{2} & 1 + \sqrt{2} \end{pmatrix} = \frac{1}{\sqrt{6}} \begin{pmatrix} -1 - \sqrt{2} & -1 + \sqrt{2} \\ 1 - \sqrt{2} & -1 - \sqrt{2} \end{pmatrix} \\ & -\frac{1}{\sqrt{6}} \begin{pmatrix} 1 - \sqrt{2} & 1 + \sqrt{2} \\ -1 - \sqrt{2} & 1 - \sqrt{2} \end{pmatrix} = \frac{1}{\sqrt{6}} \begin{pmatrix} -1 + \sqrt{2} & -1 - \sqrt{2} \\ 1 + \sqrt{2} & -1 + \sqrt{2} \end{pmatrix} \end{aligned}$$

(答)