

図チャレ 第191回 (2017年8月)

- (1) θ を $0 < \theta < 2\pi$ なる実数とするとき, $(\cos \theta + i \sin \theta) - 1$ を極形式で表せ。
- (2) $\alpha = \cos \frac{\pi}{7} + i \sin \frac{\pi}{7}$ のとき, $\frac{1}{\alpha^2 - 1}$, $\frac{1}{\alpha^4 - 1}$, $\frac{1}{\alpha^6 - 1}$ の偏角をそれぞれ求めよ。
- (3) 正7角形 $P_0P_1P_2P_3P_4P_5P_6$ に対して, $\frac{1}{P_0P_1} = \frac{1}{P_0P_2} + \frac{1}{P_0P_3}$ が成り立つことを示せ。

解答

- (1) $0 < \frac{\theta}{2} < \pi$ より $\sin \frac{\theta}{2} > 0$ であることに注意して, 2倍角の公式より

$$\begin{aligned} (\cos \theta + i \sin \theta) - 1 &= -(1 - \cos \theta) + i \sin \theta \\ &= -2 \sin^2 \frac{\theta}{2} + 2i \sin \frac{\theta}{2} \cos \frac{\theta}{2} \\ &= 2i \sin \frac{\theta}{2} \left(\cos \frac{\theta}{2} + i \sin \frac{\theta}{2} \right) \\ &= 2 \sin \frac{\theta}{2} \left(\cos \frac{\theta}{2} + i \sin \frac{\theta}{2} \right) \left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right) \\ &= 2 \sin \frac{\theta}{2} \left(\cos \frac{\theta + \pi}{2} + i \sin \frac{\theta + \pi}{2} \right) \quad (\text{答}) \end{aligned}$$

- (2) (1)を考えると,

$$\arg \frac{1}{\alpha^2 - 1} = -\arg(\alpha^2 - 1) = -\frac{1}{2} \left(\frac{2}{7}\pi + \pi \right) = -\frac{9}{14}\pi \quad (\text{答})$$

$$\arg \frac{1}{\alpha^4 - 1} = -\arg(\alpha^4 - 1) = -\frac{1}{2} \left(\frac{4}{7}\pi + \pi \right) = -\frac{11}{14}\pi \quad (\text{答})$$

$$\arg \frac{1}{\alpha^6 - 1} = -\arg(\alpha^6 - 1) = -\frac{1}{2} \left(\frac{6}{7}\pi + \pi \right) = -\frac{13}{14}\pi \quad (\text{答})$$

- (3) 複素数平面上で点 P_k を表す複素数を α^{2k} ($k = 0, 1, 2, 3, 4, 5, 6$) としてよいから,

$$\frac{1}{|\alpha^2 - 1|} = \frac{1}{|\alpha^4 - 1|} + \frac{1}{|\alpha^6 - 1|}$$

が成り立つことを示せばよい。(2)より

$$\arg \frac{1}{\alpha^2(\alpha^2 - 1)} = \arg \frac{1}{\alpha(\alpha^4 - 1)} = \arg \frac{1}{\alpha^6 - 1} = -\frac{13}{14}\pi$$

となることに注目して, 計算してみると

$$\begin{aligned}
& \frac{1}{\alpha^2(\alpha^2-1)} - \frac{1}{\alpha(\alpha^4-1)} - \frac{1}{\alpha^6-1} \\
&= \frac{(\alpha^2+1)(\alpha^4+\alpha^2+1) - \alpha(\alpha^4+\alpha^2+1) - \alpha^2(\alpha^2+1)}{\alpha^2(\alpha^2-1)(\alpha^2+1)(\alpha^4+\alpha^2+1)} \\
&= \frac{(\alpha^6+2\alpha^4+2\alpha^2+1) - (\alpha^5+\alpha^3+\alpha) - (\alpha^4+\alpha^2)}{\alpha^2(\alpha^2-1)(\alpha^2+1)(\alpha^4+\alpha^2+1)} \\
&= \frac{\alpha^6+\alpha^4+\alpha^2+1+\alpha^7(\alpha^5+\alpha^3+\alpha)}{\alpha^2(\alpha^2-1)(\alpha^2+1)(\alpha^4+\alpha^2+1)} \\
&= \frac{\alpha^{12}+\alpha^{10}+\alpha^8+\alpha^6+\alpha^4+\alpha^2+1}{\alpha^2(\alpha^2-1)(\alpha^2+1)(\alpha^4+\alpha^2+1)}
\end{aligned}$$

ここで,

$$(\alpha^2-1)(\alpha^{12}+\alpha^{10}+\alpha^8+\alpha^6+\alpha^4+\alpha^2+1) = \alpha^{14}-1=0, \quad \alpha^2-1 \neq 0$$

より

$$\alpha^{12}+\alpha^{10}+\alpha^8+\alpha^6+\alpha^4+\alpha^2+1=0$$

であるから,

$$\frac{1}{\alpha^2(\alpha^2-1)} = \frac{1}{\alpha(\alpha^4-1)} + \frac{1}{\alpha^6-1}$$

が成り立つ。両辺に $\cos \frac{11}{14}\pi + i \sin \frac{11}{14}\pi$ をかけると

$$\frac{1}{|\alpha^2(\alpha^2-1)|} = \frac{1}{|\alpha(\alpha^4-1)|} + \frac{1}{|\alpha^6-1|}$$

$|\alpha^2| = |\alpha| = 1$ より

$$\frac{1}{|\alpha^2-1|} = \frac{1}{|\alpha^4-1|} + \frac{1}{|\alpha^6-1|}$$

が導かれ,

$$\frac{1}{P_0P_1} = \frac{1}{P_0P_2} + \frac{1}{P_0P_3}$$

であることが示された。

(おわり)

(注) (3) だけであれば, 複素数の知識を用いなくて補角の公式と加法定理だけで示すこともできる。

$$\begin{aligned}
\sin \frac{\pi}{7} \left(\sin \frac{2\pi}{7} + \sin \frac{3\pi}{7} \right) &= \sin \frac{\pi}{7} \left(\sin \frac{5\pi}{7} + \sin \frac{3\pi}{7} \right) \\
&= \sin \frac{\pi}{7} \times 2 \sin \frac{4\pi}{7} \cos \frac{\pi}{7} \\
&= 2 \sin \frac{\pi}{7} \cos \frac{\pi}{7} \sin \left(\pi - \frac{3\pi}{7} \right) \\
&= \sin \frac{2\pi}{7} \sin \frac{3\pi}{7}
\end{aligned}$$

$P_0P_1 = 2 \sin \frac{\pi}{7}$, $P_0P_2 = 2 \sin \frac{2\pi}{7}$, $P_0P_3 = 2 \sin \frac{3\pi}{7}$ であるから

$$\frac{1}{2}P_0P_2 \times \frac{1}{2}P_0P_3 = \frac{1}{2}P_0P_1 \times \left(\frac{1}{2}P_0P_3 + \frac{1}{2}P_0P_2 \right)$$

$$\therefore \frac{1}{P_0P_1} = \frac{1}{P_0P_2} + \frac{1}{P_0P_3}$$

(証明おわり)